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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FRACTIONAL QUASILINEAR MIXED INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITION IN BANACH SPACES

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Abstract. In this paper, we discuss the existence and uniqueness of mild and classical solutions of quasilinear mixed integrodifferential equations of fractional orders with nonlocal condition in Banach spaces. Furthermore, we study continuous dependence of mild solutions. Our analysis is based on fractional calculus, resolvent operators and Banach's fixed point theorem.

1. Introduction

In recent years a considerable interest has been shown in the so-called fractional calculus, which allows us to consider integration and differentiation of any order, not necessarily integer. To a large extent this is due to the applications of the fractional calculus to problems in different areas of physics and engineering. The fractional calculus can be considered an old and yet novel topic. Starting from some speculations of Leibniz and Euler, followed by the works of other eminent mathematicians including Laplace, Fourier, Abel, Liouville and Riemann, it has undergone a rapid development especially during the past two decades. One of the emerging branches of this study is the theory of fractional quasilinear equations, i.e. quasilinear equations where the integer derivative with respect to time is replaced by a derivative of fractional order. The increasing interest in this class of equations is motivated both by their application to problems from viscoelasticity, heat conduction in materials with memory, electrodynamics with memory, and also because they can be employed to approach nonlinear conservation laws [1, 6, 7, 8, 9, 10, 27].

Recently, the existence of solutions of fractional abstract differential equations with nonlocal initial condition was investigated by [30]. Much attention has been paid to existence results for the nonlinear mixed integrodifferential equations with nonlocal condition in Banach spaces, see Dhakne et al. [20]. Several authors have studied the existence of solutions of abstract nonlocal problems by using different techniques, see [3, 12, 21, 25, 26, 36, 37] and the references given therein.

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Motivated by the work in [16, 20, 28, 35], we consider the quasilinear fractional integrodifferential equation

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} + A(t,x(t))x(t) = f(t,x(t),\int_0^t k(t,s,x(s))ds, \int_0^T h(t,s,x(s))ds), \quad t \in J, \tag{1.1}$$

$$x(0) + g(t_1,t_2,\ldots,t_p,x(\cdot)) = x_0, \quad x_0 \in X, \tag{1.2}$$

where $J = [0,T]$, $0 < \alpha \leq 1$ and $0 \leq t_1 < t_2 < \cdots < t_p \leq T$, $-A(t,\cdot)$ is a closed linear operator defined on a dense domain $D(A)$ in X into X such that $D(A)$ is independent of t. It is assumed also that $-A(t,\cdot)$ generates resolvent operator in the Banach space X. The nonlinear functions $f : J \times X \times X \times X \to X$, $g : J^p \times X \to X$, $k, h : J \times J \times X \to X$ are given. The operator $\frac{d^{\alpha}}{dt^{\alpha}}$ denotes the Caputo fractional derivative of order α.

In this paper our aims is to study the existence, uniqueness and other properties of solutions of the problem (1.1)–(1.2). The main tool employed in our analysis is based on the Banach fixed point theorem, resolvent operators and fractional calculus. Our results generalizes the correspondence results in [20] to nonlocal quasilinear mixed integrodifferential equations of arbitrary orders. We indicate that the definition of resolvent operators used in this paper is different from that in [16].

The rest of this article is organized as follows: In section 2 we recall briefly some basic definitions and preliminary facts which are used throughout this paper. The existence and uniqueness theorems for the problem (1.1)–(1.2) and their proofs are arranged in section 3. Finally in section 4 we give example to illustrate the application of our results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Here we recall the following known definition, for more details see [23, 29, 33].

Definition 2.1. The Riemann–Liouville fractional integral operator of order $\beta > 0$ of a function $x : [0,\infty) \to \mathbb{R}$ is defined as

$$I^{\beta}x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s)ds. \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Caputo derivative of order α, for a function $x : [0,\infty) \to \mathbb{R}$ can be written as

$$\frac{d^{\alpha}x\,(t)}{dt^{\alpha}} = \frac{1}{\Gamma\,(1-\alpha)} \int_0^t \frac{x'\,(s)}{(t-s)^{\alpha}} ds, \quad 0 < \alpha \leq 1, \tag{2.2}$$

where $x'\,(s) = \frac{dx(s)}{ds}$.

If x is an abstract function with values in X, then the integrals and derivatives which appear in (2.1) and (2.2) are taken in Bochner's sense.

Definition 2.3 (Compare[35] with [16])**.** A family of bounded linear operators $R_z(t,s) \in B(X)$, $0 \le s \le t \le T$ is called resolvent operator for equations (1.1) and (1.2) if the following conditions hold:

(a) $R_z(t,s)$ is strongly continuous in t and s, $R_z(t,t) = I$, $t \in J$.

(b) For each $x \in X$, $R_z(t,s)x$ is a continuously differentiable function in t and s such that

$$\frac{\partial^\alpha R_z}{\partial t^\alpha}(t,s)\,x = -A(t,z(t))R_z\,(t,s)\,x.$$

Here $R_z(t,s)$ can be extracted from the evolution operator of the generator $-A(t,z)$.

Next we introduce the so-called "Mild Solution" and "Classical Solution" for (1.1)–(1.2).

Definition 2.4 (Compare[35] with [16])**.** A continuous solution x of the integral equation

$$\begin{aligned} &x(t)\\ &= R_x(t,0)x_0 - R_x(t,0)g(t_1,t_2,\ldots,t_p,x(\cdot))\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}R_x(t,s)f(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h(s,\tau,x(\tau))d\tau)ds, \end{aligned} \tag{2.3}$$

with $t \in J$, is said to be a mild solution of (1.1)–(1.2) on J.

Definition 2.5 ([16, 18])**.** By a classical solution of (1.1)–(1.2) on J, we mean a function x with values in X such that:

(i) x is continuous function on J and $x(t) \in D(A)$,

(ii) $\frac{d^\alpha x}{dt^\alpha}$ exists and is continuous on $(0,T)$, and satisfying (1.1)–(1.2) on J.

Also, we need the following lemma

Lemma 2.6. [16, Lemma 3.1] *Let $\Omega \subset X$, Y be a densely and continuously imbedded Banach space in X and let $R_z(t,s)$ be the resolvent operator for the problem* (1.1)–(1.2)*, there exists a constant $C_0 > 0$ such that*

$$\|R_{z_1}(t,s)\omega - R_{z_2}(t,s)\omega\| \le C_0\|\omega\|_Y \int_s^t \|z_1(\tau) - z_2(\tau)\|d\tau,$$

for every z_1, $z_2 \in E$ with values in Ω and every $\omega \in Y$.

Now, we list the following hypotheses for our convenience. For the rest of paper, let Z be taken as both X and Y. Also, we denote by E the Banach space $C(J;X)$ of X-valued continuous functions on J equipped with the sup-norm.

(H1) There exists a constant $G > 0$ such that

$$\|g(t_1,t_2,\ldots,t_p,x_1(\cdot)) - g(t_1,t_2,\ldots,t_p,x_2(\cdot))\| \le G\|x_1 - x_2\|_E$$

for $x_1, x_2 \in E$.

(H2) There are constants L_1, K_1, H_1, G_1 and M_0 such that

$$L_1 = \max_{0\le t\le T} \|f(t,0,0,0)\|_Z,$$
$$K_1 = \max_{0\le s\le t\le T} \|k(t,s,0)\|,$$
$$H_1 = \max_{0\le s,t\le T} \|h(t,s,0)\|,$$
$$G_1 = \max_{x\in E} \|g(t_1,t_2,\dots,t_p,x(\cdot))\|_Z,$$
$$M_0 = \max_{0\le s\le t\le T} \|R_z(t,s)\|.$$

(H3) The constants $\|x_0\|, M, G_1, L, K, K_1, H, H_1, T$ and r satisfy the following two inequalities:
$[C_0\|x_0\|_Y T + M_0 G + C_0 G T] + \frac{T^{\alpha+1}C_0}{\Gamma(\alpha+1)}[Lr + LT(Kr+K_1) + LT(Hr+H_1) + L_1] + \frac{T^\alpha M_0 L}{\Gamma(\alpha+1)}[1 + KT + HT] < 1,$
and
$M_0\|x_0\| + M_0 G_1 + \frac{M_0 T^\alpha}{\Gamma(\alpha+1)}[Lr + TLKr + TLK_1 + TLHr + TLH_1 + L_1] \le r.$

With these preparations we are now in a position to state our main results to be proved in the present paper.

3. Main Results

Theorem 3.1. *Assume that*

(i) *hypotheses* (H1)–(H3) *hold,*
(ii) $f : J\times X\times X\times X \to Z$ *is continuous in* t *on* J *and there exists a constant* $L > 0$ *such that*

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\|_Z \le L(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|),$$

for $x_i, y_i, z_i \in B_r$, $i = 1,2$, *where* $B_r = \{x\in X : \|x\| \le r\}$.
(iii) $k, h : J\times J\times X \to X$ *are continuous in* s,t *on* J *and there exist positive constants* K, H *such that*

$$\|k(t,s,x_1) - k(t,s,x_2)\| \le K(\|x_1-x_2\|),$$
$$\|h(t,s,x_1) - h(t,s,x_2)\| \le H(\|x_1-x_2\|),$$

for $x_i, y_i \in B_r$, $i = 1,2$.

Then problem (1.1)–(1.2) *has a unique mild solution on* J.

Proof of Theorem 3.1. We shall use the notions and notations introduced in the preceding section. We define an operator $F : E \to E$ by

$$\begin{aligned}(Fz)(t)& \\ &= R_z(t,0)x_0 - R_z(t,0)g(t_1,t_2,\dots,t_p,z(\cdot)) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}R_z(t,s)f(s,z(s),\int_0^s k(s,\tau,z(\tau))d\tau,\int_0^T h(s,\tau,z(\tau))d\tau)ds,\end{aligned} \tag{3.1}$$

for $t\in J$. It follows from assumption on the functions f, h and k that $F : E\to E$ and for every $z\in E$, $Fz(0) = x_0 - g(t_1,t_2,\dots,t_p,z(\cdot))$.

Let S be the nonempty closed and bounded set given by

$$S = \{z \in E : z(0) = x_0 - g(t_1, t_2, \dots, t_p, z(\cdot)),\ \|z(t)\| \leq r\}. \tag{3.2}$$

Then for $z \in S$ we have

$$\begin{aligned}
&\|(Fz)(t)\| \\
&\leq \|R_z(t,0)x_0\| - \|R_z(t,0)g(t_1, t_2, \dots, t_p, z(\cdot))\| \\
&\quad + \|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} R_z(t,s) f(s, z(s), \int_0^s k(s,\tau,z(\tau))d\tau, \int_0^T h(s,\tau,z(\tau))d\tau)ds\| \\
&\leq M_0\|x_0\| + M_0 G_1 + \frac{M_0}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\|f(s, z(s), \int_0^s k(s,\tau,z(\tau))d\tau, \\
&\quad \int_0^T h(s,\tau,z(\tau))d\tau) - f(s,0,0,0)\| + \|f(s,0,0,0)\|]ds \\
&\leq M_0\|x_0\| + M_0 G_1 + \frac{M_0}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[L(\|z(s) - 0\| + \|\int_0^s k(s,\tau,z(\tau))d\tau - 0\| \\
&\quad + \|\int_0^T h(s,\tau,z(\tau))d\tau - 0\|) + \|f(s,0,0,0)\|]ds \\
&\leq M_0\|x_0\| + M_0 G_1 \\
&\quad + \frac{M_0}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Lr + L\int_0^s \|k(s,\tau,z(\tau)) - k(s,\tau,0) + k(s,\tau,0)\|d\tau \\
&\quad + L\int_0^T \|h(s,\tau,z(\tau)) - h(s,\tau,0) + h(s,\tau,0)\|d\tau + L_1]ds \\
&\leq M_0\|x_0\| + M_0 G_1 \\
&\quad + \frac{M_0}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1]ds \\
&\leq M_0\|x_0\| + M_0 G_1 \\
&\quad + \frac{M_0 T^\alpha}{\Gamma(\alpha+1)}[Lr + TLKr + TLK_1 + TLHr + TLH_1 + L_1] \leq r.
\end{aligned}$$

Thus, we have $F : S \to S$.

Now, for every $z_1, z_2 \in S$ and $t \in J$, we have

$$\begin{aligned}
&\|(Fz_1)(t) - (Fz_2)(t)\| \\
&\leq \|R_{z_1}(t,0)x_0 - R_{z_2}(t,0)x_0\| \\
&\quad + \|R_{z_1}(t,0)g(t_1, t_2, \dots, t_p, z_1(\cdot)) - R_{z_2}(t,0)g(t_1, t_2, \dots, t_p, z_2(\cdot))\| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad \times [\|R_{z_1}(t,0) f(s, z_1(s), \int_0^s k(s,\tau,z_1(\tau))d\tau, \int_0^T h(s,\tau,z_1(\tau))d\tau) \\
&\quad -R_{z_2}(t,0) f(s, z_2(s), \int_0^s k(s,\tau,z_2(\tau))d\tau, \int_0^T h(s,\tau,z_2(\tau))d\tau)\|]ds
\end{aligned}$$

$$
\begin{aligned}
&\le \|R_{z_1}(t,0)x_0 - R_{z_2}(t,0)x_0\| \\
&\quad + \|R_{z_1}(t,0)g(t_1,t_2,\dots,t_p,z_1(\cdot)) - R_{z_2}(t,0)g(t_1,t_2,\dots,t_p,z_2(\cdot))\| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad \times \Bigg[\|R_{z_1}(t,0)f(s,z_1(s),\int_0^s k(s,\tau,z_1(\tau))d\tau,\int_0^T h(s,\tau,z_1(\tau))d\tau) \\
&\quad - R_{z_2}(t,0)f(s,z_1(s),\int_0^s k(s,\tau,z_1(\tau))d\tau,\int_0^T h(s,\tau,z_1(\tau))d\tau)\| \\
&\quad + \|R_{z_2}(t,0)f(s,z_1(s),\int_0^s k(s,\tau,z_1(\tau))d\tau,\int_0^T h(s,\tau,z_1(\tau))d\tau) \\
&\quad -R_{z_2}(t,0)f(s,z_2(s),\int_0^s k(s,\tau,z_2(\tau))d\tau,\int_0^T h(s,\tau,z_2(\tau))d\tau)\|\Bigg]ds \\
&\le I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \|R_{z_1}(t,0)x_0 - R_{z_2}(t,0)x_0\| \\
&\quad + \|R_{z_1}(t,0)g(t_1,t_2,\dots,t_p,z_1(\cdot)) - R_{z_2}(t,0)g(t_1,t_2,\dots,t_p,z_2(\cdot))\|, \\
I_2 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad \times [\|R_{z_1}(t,0)f(s,z_1(s),\int_0^s k(s,\tau,z_1(\tau))d\tau,\int_0^T h(s,\tau,z_1(\tau))d\tau) \\
&\quad - R_{z_2}(t,0)f(s,z_1(s),\int_0^s k(s,\tau,z_1(\tau))d\tau,\int_0^T h(s,\tau,z_1(\tau))d\tau)\|]ds,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad + [\|R_{z_2}(t,0)f(s,z_1(s),\int_0^s k(s,\tau,z_1(\tau))d\tau,\int_0^T h(s,\tau,z_1(\tau))d\tau) \\
&\quad - R_{z_2}(t,0)f(s,z_2(s),\int_0^s k(s,\tau,z_2(\tau))d\tau,\int_0^T h(s,\tau,z_2(\tau))d\tau)\|]ds.
\end{aligned}
$$

Using Lemma (2.6) and hypotheses (H1), (H2), we obtain

$$
\begin{aligned}
I_1 &\le \|R_{z_1}(t,0)x_0 - R_{z_2}(t,0)x_0\| \\
&\quad + \|R_{z_1}(t,0)g(t_1,t_2,\dots,t_p,z_1(\cdot)) - R_{z_1}(t,0)g(t_1,t_2,\dots,t_p,z_2(\cdot))\| \\
&\quad + \|R_{z_1}(t,0)g(t_1,t_2,\dots,t_p,z_2(\cdot)) - R_{z_2}(t,0)g(t_1,t_2,\dots,t_p,z_2(\cdot))\| \\
&\le C_0\|x_0\|_Y \int_0^t \|z_1(\tau) - z_2(\tau)\|d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ \|R_{z_1}(t,0)\| \|g(t_1,t_2,\ldots,t_p,z_1(\cdot)) - g(t_1,t_2,\ldots,t_p,z_2(\cdot))\| \\
&+ C_0 \|g(t_1,t_2,\ldots,t_p,z_2(\cdot))\|_Y \int_0^t \|z_1(\tau) - z_2(\tau)\| d\tau \\
\leq\ & C_0 \|x_0\|_Y \|z_1 - z_2\|_E \int_0^t d\tau + M_0 G \|z_1 - z_2\|_E \\
&+ C_0 \|g(t_1,t_2,\ldots,t_p,z_2(\cdot))\|_Y \|z_1 - z_2\|_E \int_0^t d\tau
\end{aligned}
$$

Thus

$$
I_1 \leq [C_0 \|x_0\|_Y T + M_0 G + C_0 G T] \|z_1 - z_2\|_E. \tag{3.3}
$$

Applying Lemma (2.6), hypotheses (H2), and assumptions (ii), (iii), we get

$$
\begin{aligned}
I_2 \leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_0 \int_0^t \|z_1(\tau) - z_2(\tau)\| d\tau \\
& \times [\|f(s, z_1(s), \int_0^s k(s,\tau,z_1(\tau)) d\tau, \int_0^T h(s,\tau,z_1(\tau)) d\tau) - f(s,0,0,0)\|_Y \\
& + \|f(s,0,0,0)\|_Y] ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_0 \|z_1 - z_2\|_E \int_0^t d\tau \\
& \times [L(\|z(s) - 0\| + \| \int_0^s k(s,\tau,z_1(\tau)) d\tau - 0\| \\
& + \| \int_0^T h(s,\tau,z_1(\tau)) d\tau - 0\|) + \|f(s,0,0,0)\|_Y] ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_0 \|z_1 - z_2\|_E T \\
& \times [Lr + L \int_0^s \|k(s,\tau,z_1(\tau)) - k(s,\tau,0) + k(s,\tau,0)\| d\tau \\
& + L \int_0^T \|h(s,\tau,z_1(\tau)) - h(s,\tau,0) + h(s,\tau,0)\| d\tau + L_1] ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_0 \|z_1 - z_2\|_E T \\
& \times [Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] ds \\
\leq\ & \frac{T^{\alpha+1} C_0}{\Gamma(\alpha+1)} [Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] \|z_1 - z_2\|_E,
\end{aligned}
$$

$$
I_2 \leq \frac{T^{\alpha+1} C_0}{\Gamma(\alpha+1)} [Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] \|z_1 - z_2\|_E. \tag{3.4}
$$

Again by using Lemma (2.6), hypotheses (H2), and assumptions (ii), (iii), we obtain

$$
I_3 \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|R_{z_2}(t,0)\|
$$

$$
\begin{aligned}
&\times [\|f(s, z_1(s), \int_0^s k(s,\tau,z_1(\tau))d\tau, \int_0^T h(s,\tau,z_1(\tau))d\tau) \\
&- f(s, z_2(s), \int_0^s k(s,\tau,z_2(\tau))d\tau, \int_0^T h(s,\tau,z_2(\tau))d\tau)\|]ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_0 L [\|z_1(s) - z_2(s)\| \\
&+ \int_0^s \|k(s,\tau,z_1(\tau)) - k(s,\tau,z_2(\tau))\| d\tau \\
&+ \int_0^T \|h(s,\tau,z_1(\tau)) - h(s,\tau,z_2(\tau))\| d\tau] ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_0 L \|z_1 - z_2\|_E [1 + K \int_0^s d\tau + H \int_0^T d\tau] ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_0 L \|z_1 - z_2\|_E [1 + KT + HT] ds,
\end{aligned}
$$

$$
I_3 \leq \frac{T^\alpha M_0 L}{\Gamma(\alpha+1)} [1 + KT + HT] \|z_1 - z_2\|_E. \tag{3.5}
$$

Hence from (3.3)–(3.5), we have

$$
\|Fz_1 - Fz_2\|_E \leq q \|z_1 - z_2\|_E,
$$

where $q = [C_0\|x_0\|_Y T + M_0 G + C_0 G T] + \frac{T^{\alpha+1} C_0}{\Gamma(\alpha+1)}[Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] + \frac{T^\alpha M_0 L}{\Gamma(\alpha+1)}[1 + KT + HT]$, with $0 < q < 1$. Thus F is a strict contraction map from S into S and therefore by Banach contraction principle there exists unique fixed point x of F in S and this point is the mild solution of problem (1.1)–(1.2) on J. This completes the proof of the Theorem 3.1. □

To establish the existence of unique classical solution to (1.1)–(1.2), we shall require the following lemma.

Lemma 3.2. *Assume that $|\tilde{t}_2 - \tilde{t}_1| \leq 1$ and $0 < \alpha \leq 1$. Then, there exists a constant $N_0 > 0$ such that*

$$
\|[R_z(\tilde{t}_2, s) - R_z(\tilde{t}_1, s)]x\| \leq N_0 \|x\| |\tilde{t}_2 - \tilde{t}_1|^\alpha \text{ for every } x, z \in D(A). \tag{3.6}
$$

Proof. It follows from (b) of Definition 2.3 that $R_z(t,s)x$ is continuously differentiable in $t \in J$. Using mean value theorem for derivatives, we obtain

$$
\begin{aligned}
\|[R_z(\tilde{t}_2, s) - R_z(\tilde{t}_1, s)]x\| &\leq \sup_{t \in J} \left\| \frac{\partial R_z}{\partial t}(t,s)x \right\| |\tilde{t}_2 - \tilde{t}_1| \\
&\leq N_0 \|x\| |\tilde{t}_2 - \tilde{t}_1| \\
&\leq N_0 \|x\| |\tilde{t}_2 - \tilde{t}_1|^\alpha,
\end{aligned} \tag{3.7}
$$

where $|\tilde{t}_2 - \tilde{t}_1| \leq 1$, $0 < \alpha \leq 1$ and $\sup_{t \in J} \left\| \frac{\partial R_z}{\partial t}(t,s)x \right\| \leq N_0 \|x\|$ for some $N_0 > 0$.

□

Theorem 3.3. *Assume that*

(i) *hypotheses* (H1)–(H3) *hold,*

(ii) X *is a reflexive Banach space with norm* $\|\cdot\|$ *and* $x_0 \in D(A)$*, the domain of* $A(t,\cdot)$*,*

(iii) $g(t_1, t_2, \ldots, t_p, x(\cdot)) \in D(A)$*,*

(iv) *There exists a constant* $L > 0$ *such that*

$$\begin{aligned}\|f(t_1, x_1, y_1, z_1) - f(t_2, x_2, y_2, z_2)\|_Z \le L(&|t_1 - t_2| + \|x_1 - x_2\| + \|y_1 - y_2\| \\ &+ \|z_1 - z_2\|),\end{aligned}$$

(v) *There exist constants* $K, H > 0$ *such that*

$$\begin{aligned}\|k(t_1, s, x_1) - k(t_2, s, x_2)\| &\le K(|t_1 - t_2| + \|x_1 - x_2\|), \\ \|h(t_1, s, x_1) - h(t_2, s, x_2)\| &\le H(|t_1 - t_2| + \|x_1 - x_2\|),\end{aligned}$$

Then x *is a unique classical solution of* (1.1)–(1.2) *on* J*.*

Proof of Theorem 3.3. All the assumptions of Theorem 3.1 are being satisfied, then problem (1.1)–(1.2) has a unique mild solution belonging to S and given by

$$\begin{aligned} &x(t) \\ &= R_x(t,0)x_0 - R_x(t,0)g(t_1, t_2, \ldots, t_p, x(\cdot)) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} R_x(t,s) f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^T h(s,\tau,x(\tau))d\tau)ds. \end{aligned} \tag{3.8}$$

Since J is compact it is easy to check that x is Hölder continuous on J if it is locally Hölder continuous. Now we will show that x is locally Hölder continuous.

For simplification, set

$$\tilde{f}(t) = f(t, x(t), \int_0^t k(t,s,x(s))ds, \int_0^T h(t,s,x(s))ds). \tag{3.9}$$

Then (3.8) can be written as

$$x(t) = R_x(t,0)x_0 - R_x(t,0)g(t_1, t_2, \ldots, t_p, x(\cdot)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} R_x(t,s)\tilde{f}(s)ds. \tag{3.10}$$

Since x is continuous on J and the map f satisfy the assumptions (iv) and (v), it fellows that $\tilde{f}$ is continuous, and therefore bounded on J, set $N_1 := \sup_{t \in J}\|\tilde{f}(t)\|$.

Next, let $t \in J$ be fixed and let $\tilde{t}_1$, $\tilde{t}_2$ be in $(t-\delta, t+\delta)$ with $\tilde{t}_1 \leq \tilde{t}_2$ and $\delta > 0$, we have

$$\begin{aligned} & x(\tilde{t}_2) - x(\tilde{t}_1) \\ & = [R_x(\tilde{t}_2, 0) - R_x(\tilde{t}_1, 0)]x_0 - [R_x(\tilde{t}_2, 0) - R_x(\tilde{t}_1, 0)]g(t_1, t_2, \dots, t_p, x(\cdot)) \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{\tilde{t}_1}^{\tilde{t}_2} (\tilde{t}_2 - s)^{\alpha-1} R_x(\tilde{t}_2, s)\tilde{f}(s) ds \\ & \quad - \frac{1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} [(\tilde{t}_1 - s)^{\alpha-1} - (\tilde{t}_2 - s)^{\alpha-1}] R_x(\tilde{t}_2, s)\tilde{f}(s) ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} (\tilde{t}_1 - s)^{\alpha-1} [R_x(\tilde{t}_2, s) - R_x(\tilde{t}_1, s)]\tilde{f}(s) ds. \end{aligned} \tag{3.11}$$

$$\|x(\tilde{t}_2) - x(\tilde{t}_1)\| \leq \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4. \tag{3.12}$$

Using Lemma (3.2) for a small enough $\delta > 0$, we get

$$\begin{aligned} \tilde{I}_1 & \leq \|[R_x(\tilde{t}_2, 0) - R_x(\tilde{t}_1, 0)]x_0 - [R_x(\tilde{t}_2, 0) - R_x(\tilde{t}_1, 0)]g(t_1, t_2, \dots, t_p, x(\cdot))\| \\ & \leq [N_0\|x_0\| + N_0 G_1]|\tilde{t}_2 - \tilde{t}_1|^{\alpha}, \end{aligned} \tag{3.13}$$

for $\tilde{I}_2$, we have

$$\begin{aligned} \tilde{I}_2 & \leq \frac{1}{\Gamma(\alpha)} \int_{\tilde{t}_1}^{\tilde{t}_2} (\tilde{t}_2 - s)^{\alpha-1} \|R_x(\tilde{t}_2, s)\tilde{f}(s)\| ds \\ & \leq \frac{M_0 N_1}{\Gamma(\alpha+1)} |\tilde{t}_2 - \tilde{t}_1|^{\alpha}, \end{aligned} \tag{3.14}$$

and for $\tilde{I}_3$, we have

$$\begin{aligned} \tilde{I}_3 & \leq \frac{1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} |(\tilde{t}_1 - s)^{\alpha-1} - (\tilde{t}_2 - s)^{\alpha-1}| \|R_x(\tilde{t}_2, s)\tilde{f}(s)\| ds \\ & \leq \frac{M_0 N_1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} |(\tilde{t}_1 - s)^{-\mu} - (\tilde{t}_2 - s)^{-\mu}| ds, \end{aligned} \tag{3.15}$$

with $\mu = 1 - \alpha$. Here we can use the calculation presented in [31, Theorem 3.2] to find the upper bound of integral and thus we get

$$\tilde{I}_3 \leq \frac{M_0 N_1}{\Gamma(\alpha)} \mu \delta_1^{\mu-1} (1-c)^{(\mu-1)-1} |\tilde{t}_2 - \tilde{t}_1|^{1-\mu}, \tag{3.16}$$

where $c = (1-\mu)^{\frac{1}{\mu}}$ and $0 < \delta_1 \leq 1$.

Using again (3.6), we may calculate the bound of $\tilde{I}_4$ as

$$
\begin{aligned}
\tilde{I}_4 &\le \frac{1}{\Gamma(\alpha)}\int_0^{\tilde{t}_1}(\tilde{t}_1-s)^{\alpha-1}\|[R_x(\tilde{t}_2,s)-R_x(\tilde{t}_1,s)]\tilde{f}(s)\|ds\\
&\le \frac{N_0\|\tilde{f}(s)\|}{\Gamma(\alpha)}\int_0^{\tilde{t}_1}(\tilde{t}_1-s)^{\alpha-1}|\tilde{t}_2-\tilde{t}_1|^{\alpha}ds\\
&\le \frac{T^{\alpha}}{\Gamma(\alpha+1)}N_0\|\tilde{f}(s)\||\tilde{t}_2-\tilde{t}_1|^{\alpha}\\
&\le \frac{T^{\alpha}}{\Gamma(\alpha+1)}N_0N_1|\tilde{t}_2-\tilde{t}_1|^{\alpha},
\end{aligned}
\tag{3.17}
$$

Hence from (3.13)–(3.17), locally Hölder continuity of $x(t)$ follows.

As pointed out earlier in this proof, we may deduce that $x(t)$ is Hölder continuous on J. The Hölder continuity of $x(t)$ on J combined with (iv) and (v) of Theorem (3.3) implies $\tilde{f}(t)$ is Hölder continuous on J. According to [16, Theorem 3.4], we observe that the equation

$$
\frac{d^{\alpha}y(t)}{dt^{\alpha}}+A(t,y(t))y(t)=f(t,x(t),\int_0^t k(t,s,x(s))ds,\int_0^T h(t,s,x(s))ds),\quad t\in J
$$
$$
y(0)=x_0-g(t_1,t_2,\dots,t_p,x(\cdot))
$$

has a unique classical solution $y(t)$ on J satisfying the equation

$$
\begin{aligned}
&y(t)\\
&=R_x(t,0)x_0-R_x(t,0)g(t_1,t_2,\dots,t_p,x(\cdot))\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}R_x(t,s)f(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau,\int_0^T h(s,\tau,x(\tau))d\tau)ds\\
&=x(t),\quad t\in J.
\end{aligned}
$$

Consequently, $x(t)$ is the classical solution of initial value problem (1.1)–(1.2) on J. This completes the proof of Theorem 3.3. □

The following generalized Gronwall's inequality is essential to prove continuous dependence of mild solutions of equations (1.1)–(1.2)

Lemma 3.4. [24] *Suppose $b\ge 0$, $\beta>0$ and $a(t)$ is a nonnegative function locally integrable on $0\le t<T$ (some $T\le\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0\le t<T$ with*

$$
u(t)\le a(t)+b\int_0^t(t-s)^{\beta-1}u(s)ds,\quad 0\le t<T.
$$

Then

$$
u(t)\le a(t)+\int_0^t\sum_{j=1}^{\infty}\frac{(b\Gamma(\beta))^j}{\Gamma(j\beta)}(t-s)^{j\beta-1}a(s)ds,\quad 0\le t<T.
\tag{3.18}
$$

If $a(t)\equiv a$, constant on $0\le t<T$, then the inequality (3.18) is reduced to

$$
u(t)\le aE_{\beta}(b\Gamma(\beta)t^{\beta}),
$$

where E_{β} is the Mittag–Leffler function of order β.

Theorem 3.5. *Suppose that the functions f, g, k and h satisfy hypotheses* (H1)-(H4) *and assumptions (ii), (iii) of Theorem 3.1. Then, for each pair of elements $x_0^*, x_0^{**} \in X$, and for the corresponding mild solutions x_1, x_2 of problem* (1.1) *with $x_1(t_0) + g(t_1, t_2, \ldots, t_p, x_1(\cdot)) = x_0^*$ and $x_2(t_0) + g(t_1, t_2, \ldots, t_p, x_2(\cdot)) = x_0^{**}$, the inequality*

$$\|x_1 - x_1\|_E \leq \frac{M_0}{(1-p_1)} \|x_0^* - x_0^{**}\| E_\alpha\left(\frac{p_2}{(1-p_1)} \Gamma(\alpha) t^\alpha\right),$$

is true, whenever

$$\begin{aligned} p_1 &= \frac{T^{\alpha+1} C_0}{\Gamma(\alpha+1)} [Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] \\ &\quad + [C_0 \|x_0\|_Y T + M_0 G + C_0 G T] \\ &< 1 \end{aligned}$$

and

$$p_2 = M_0 L [1 + KT + HT].$$

Proof of Theorem 3.5. Suppose that $x_1(t)$ and $x_2(t)$ satisfy (1.1) on J with $x_1(t_0) + g(t_1, t_2, \ldots, t_p, x_1(\cdot)) = x_0^*$ and $x_2(t_0) + g(t_1, t_2, \ldots, t_p, x_2(\cdot)) = x_0^{**}$, respectively and $x_1, x_2 \in E$. Using the equation (2.3), hypotheses (H1)–(H4) and assumptions (ii), (iii), we obtain

$$\begin{aligned} &\|x_1(t) - x_2(t)\| \\ &\leq \|R_{x_1}(t,0) x_0^* - R_{x_2}(t,0) x_0^{**}\| \\ &\quad + \|R_{x_1}(t,0) g(t_1, t_2, \ldots, t_p, x_1(\cdot)) - R_{x_2}(t,0) g(t_1, t_2, \ldots, t_p, x_2(\cdot))\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \Bigg[\|R_{x_1}(t,0) f(s, x_1(s), \int_0^s k(s,\tau,x_1(\tau)) d\tau, \int_0^T h(s,\tau,x_1(\tau)) d\tau) \\ &\quad - R_{x_2}(t,0) f(s, x_2(s), \int_0^s k(s,\tau,x_2(\tau)) d\tau, \int_0^T h(s,\tau,x_2(\tau)) d\tau) \| ds \Bigg] \\ &\leq \|R_{x_1}(t,0) x_0^* - R_{x_2}(t,0) x_0^*\| + \|R_{x_2}(t,0) x_0^* - R_{x_2}(t,0) x_0^{**}\| \\ &\quad + \|R_{x_1}(t,0) g(t_1, t_2, \ldots, t_p, x_1(\cdot)) - R_{x_1}(t,0) g(t_1, t_2, \ldots, t_p, x_2(\cdot))\| \\ &\quad + \|R_{x_1}(t,0) g(t_1, t_2, \ldots, t_p, x_2(\cdot)) - R_{x_2}(t,0) g(t_1, t_2, \ldots, t_p, x_2(\cdot))\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \Bigg[\|R_{x_1}(t,0) f(s, x_1(s), \int_0^s k(s,\tau,x_1(\tau)) d\tau, \int_0^T h(s,\tau,x_1(\tau)) d\tau) \\ &\quad - R_{x_2}(t,0) f(s, x_1(s), \int_0^s k(s,\tau,x_1(\tau)) d\tau, \int_0^T h(s,\tau,x_1(\tau)) d\tau) \| ds \end{aligned}$$

$$+\|R_{x_2}(t,0)f(s,x_1(s),\int_0^s k(s,\tau,x_1(\tau))d\tau,\int_0^T h(s,\tau,x_1(\tau))d\tau)$$
$$\left.-R_{x_2}(t,0)f(s,x_2(s),\int_0^s k(s,\tau,x_2(\tau))d\tau,\int_0^T h(s,\tau,x_2(\tau))d\tau)\|\right] ds$$

Now, we can use the same calculation presented in proof of Theorem (3.1) to find

$$\begin{aligned}\|x_1(t)-x_2(t)\| &\le M_0\|x_0^*-x_0^{**}\|+p_1\|x_1-x_2\|_E\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p_2\|x_1-x_2\|_E ds,\end{aligned}$$

where

$$\begin{aligned}p_1 = &\frac{T^{\alpha+1}C_0}{\Gamma(\alpha+1)}[Lr+LT(Kr+K_1)+LT(Hr+H_1)+L_1]\\ &+[C_0\|x_0\|_Y T+M_0G+C_0GT],\end{aligned}$$

and

$$p_2 = M_0L[1+KT+HT].$$

Therefore, we obtain

$$\begin{aligned}\|x_1-x_2\|_E \le &\frac{M_0}{(1-p_1)}\|x_0^*-x_0^{**}\|\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{p_2}{(1-p_1)}\|x_1-x_2\|_E ds.\end{aligned}$$

Using Lemma (3.4), we get

$$\|x_1-x_2\|_E \le \frac{M_0}{(1-p_1)}\|x_0^*-x_0^{**}\|E_\alpha(\frac{p_2}{(1-p_1)}\Gamma(\alpha)t^\alpha),$$

provided that $p_1<1$. From this inequality, it follows that the continuous dependence of solutions depends upon the initial data. This completes the proof of the Theorem 3.5. □

4. Application

In this section we present an example to illustrate the applications of some of our main results, we consider the fractional mixed Volterra–Fredholm partial integrodifferential equation

$$\begin{aligned}&\frac{\partial^\alpha w(u,t)}{\partial t^\alpha}+a(u,t,w(u,t))\frac{\partial^2 w(u,t)}{\partial u^2}\\ &=P(t,w(u,t),\int_0^t k_1(t,s,w(u,s))ds,\int_0^T h_1(t,s,w(u,s))ds)\\ &\quad 0<u<1,\quad 0\le t\le T\end{aligned}\tag{4.1}$$

with initial and boundary conditions

$$w(0,t)=w(1,t)=0,\quad 0\le t\le T,\tag{4.2}$$

$$w(u,0)+\sum_{i=1}^{p} w(u,t_i)=w_0(u), \quad 0<t_1<t_2<\cdots<t_p\le T. \tag{4.3}$$

where $a:(0,1)\times[0,T]\times\mathbb{R}\to\mathbb{R}$, $P:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $k_1,h_1:[0,T]\times[0,T]\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

First, we reduce the equations (4.1)–(4.3) into (1.1)–(1.2) by making suitable choices of A, f, g, k and h.

Let $X=L^2[0,1]$ be the space of square integrable functions. Define the operator $A(t,\cdot):X\to X$ by $(A(t,\cdot)z)(u)=a(u,t,\cdot)z''$ with dense domain $D(A(t,\cdot))=\{z\in X: z,z'$ are absolutely continuous, $z''\in X$ and $z(0)=z(1)=0\}$, generates an evolution system and $R_x(t,s)$ can be extracted from evolution system, such that $\|R_x(t,s)\|\le M_0$, $M_0>0$ for $s<t$ and $x\in\Omega\subset X$ (see [16, 17, 34]).

Define the functions $f:[0,T]\times X\times X\times X\to X$, $k:[0,T]\times[0,T]\times X\to X$, $h:[0,T]\times[0,T]\times X\to X$ and $g:[0,T]^p\times X\to X$ as follows

$$\begin{aligned} f(t,x,y,z)(u) &= P(t,x(u),y(u),z(u)),\\ k(t,s,x)(u) &= k_1(t,s,x(u)),\\ h(t,s,x)(u) &= h_1(t,s,x(u)),\\ g(t_1,t_2,\dots,t_p,x(\cdot))u &= \sum_{i=1}^{p} w(u,t_i) \end{aligned}$$

for $t\in[0,T], x,y,z\in X$ and $0<u<1$. We assume that the functions P, k_1 and h_1 in (4.1) satisfy all the hypotheses of the Theorem 3.1. Also we suppose that

$$|\sum_{i=1}^{p} w(u,t_i)-\sum_{i=1}^{p} w(v,t_i)|\le G^*\sup_{t\in[0,T]}|u(t)-v(t)|$$

for $u,v\in E_1=C([0,T];\mathbb{R})$ and some constant $G^*>0$. Then the above problem (4.1)–(4.3) can be formulated abstractly as quasilinear mixed integrodifferential equation in Banach space X:

$$\frac{\partial^\alpha x(t)}{\partial t^\alpha}+A(t,x(t))x(t)=f(t,x(t),\int_0^t k(t,s,x(s))ds,\int_0^T h(t,s,x(s))ds), \quad t\in J \tag{4.4}$$

$$x(t_0)+g(t_1,t_2,\dots,t_p,x(\cdot))=x_0. \tag{4.5}$$

Since all the hypotheses of the Theorem 3.1 are satisfied, the Theorem 3.1 can be applied to guarantee the mild solution of the fractional mixed Volterra–Fredholm partial integrodifferential equations (4.1)–(4.3).

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